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The Impact of Damping and Fractinality in Nonlinear Systems: Solving the Fractional Damped Burgers Equation Using the Coupled Local Fractional Elzaki Adomian Decomposition Method

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Abstract

This study explores the application of the Fractional Elzaki Adomian Decomposition Method (LFEADM) to the Fractional Damped Burgers Equation, emphasizing the effects of varying damping factors ξ on system dynamics. EADM combines the Elzaki Transform and Adomian Decomposition Method to efficiently handle fractional derivatives and nonlinear terms, transforming the equation into a simpler form. The results reveal that damping influences wave propagation, while zero damping yields behavior that looks exactly like the classical Burgers equation, with solutions that match those found in previous studies.

Keywords: Damp, Burger, Adomian, Polynomial, Fractional, Integer-order

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Introduction

The Fractional Damped Burgers Equation (FDBE) began from the problems brought by the addition of damping with integer order into the Burgers equation. (Abd et al. 2021, Abd et al. 2018). Damping is an important part that is considered in the control of the behaviour of nonlinear systems. It represents a loss of energy from the system considered. For example, damping can represent the force of friction that go against fluid flow. in other systems, it can simulate signal strength deterioration or kinetic energy loss in mechanical systems (Huang et al. 2019, Al-Hababi et al 2020).

Fractional Damped Burgers Equation integrates fractional order and damping into a single equation. The nonlinear structure and fractional derivatives of this equation makes it difficult to solve analytically. Especially for some boundary conditions, traditional techniques of solving such equations can pose a problem (Guo, 2015, Liu & Zhang, 2023, Olubanwo et al. 2023). An efficient method for decomposing nonlinear terms in an equations is the Adomian Decomposition Method (Sanchez, 2011, Olubanwo et al 2024). Elzaki Transform also offers a strong integral transform that makes working with fractional derivatives easier (Elzaki, 2011, Sharjeel et al, 2019).

This work examines, the effects of integer order on the damped nonlinear system. The primary aim is to illustrate the solution of this problem by using the coupled Elzaki-Adomian decomposition method (EADM). Our goal is to provide an approximation solution similar to it exact solution, revealing the influence of the fractional order on the damped coefficients of the system's behaviour.

Materials and Methods

Elzaki Adomian Decomposition Method is applied to the Fractional Damped Burgers Equation to solve nonlinear fractional differential equations with damping effects.

Fractional Damped Burgers Equation

The Fractional Damped Burgers Equation (FDBE) is an extension of the l Burgers equation, combing both fractional order derivatives and damping to capture the effects of dissipation in Burgers equation. The classical Burgers equation is written as (Olubanwo et al. 2018):

$$\frac{\partial u}{\partial t}+\frac{u\partial u}{\partial x}=\frac{v\partial^2 u}{\partial x^2} \qquad (1)$$

Where$u(x,t)$ is the unknown variable,v is the diffusion coefficient$x$ and t are both independent spatial and time variable respectively.

The Fractional Damped Burgers Equation is written as (Djelloul 2021):

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{u\partial u}{\partial x}=\frac{v\partial^2 u}{\partial x^2}-\xi u(x,t) \qquad (2)$$

Where

- $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the fractional time derivative of order α
- $u\frac{\partial u}{\partial x}$ is the nonlinear advection term,
- $\frac{v\partial^2 u}{\partial x^2}$ is the diffusion term,
- $-\xi(x,t)$ is the damping term

Elzaki Transform of Fractional Order

Definition 1. The Fractional Elzaki Transform of $f(x)$ of order α is (Djelloul 2021)

$$E_\alpha\{f(t)\}=T_\alpha(v)=\frac{1}{\Gamma(1+\alpha)}v^\alpha\int_0^\infty E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)f(t)(dt)^\alpha,$$

$$0<\alpha\le 1 \qquad (3)$$

The inverse transform is obtained using

$$E_\alpha^{-1}\{T_{\alpha(v)}\}=f(x) \qquad (4)$$

$$E_\alpha[f(t)]=E_\alpha\left\{\sum_{n=0}^{\infty}g_n t^{n\alpha}\right\}=\sum_{n=0}^{\infty}\Gamma(1+n\alpha)g_n v^{n\alpha+2\alpha}, \qquad (5)$$

Properties of Elzaki Transform

Theorem 1.

If $E_\alpha\{q(t)\}=Q_\alpha(v)$ and $E_\alpha\{h(t)\}=H_\alpha(v)$, then one has

$$E_\alpha\{\lambda q(t)+\mu h(t)\}=\lambda Q_\alpha(v)+\mu H_\alpha(v) \qquad (6)$$

Where λ and μ are arbitrary constants

Proof:

$$E_\alpha\{\lambda q(t)+\mu h(t)\}=\frac{v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)\{\lambda q(t)+\mu h(t)\}(dt)^\alpha, \qquad (7)$$

$$=\frac{v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty\left[E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)(\lambda q(t))+E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)(\mu h(t))\right](dt)^\alpha, \qquad (8)$$

$$=\frac{\lambda v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty\left[E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)(\lambda qt)\right](dt)^\alpha+\frac{\mu v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty\left[E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)(\lambda h(t))\right](dt)^\alpha, \qquad (9)$$

$$=\lambda E_\alpha\{q(t)\}+\mu E_\alpha\{h(x)\}$$

$$=\lambda Q_\alpha(v)+\mu H_\alpha(v) \qquad (10)$$

This ends the proof

Theorem 2:

If $E_\alpha\{f(x)\}=T_\alpha(v)$, so

$$E_\alpha\{D_{0+}^\alpha f(t)\}=\frac{1}{v^\alpha}T_{\alpha(v)}-v^\alpha f(0),\quad 0<\alpha\le 1 \quad (11)$$

and

$$E_\alpha\{D_{0+}^{n\alpha}f(t)\} = \frac{1}{v^{n\alpha}}T_\alpha(v) - \sum_{k=0}^{n-1} v^{(k-n+2)\alpha} f^{(k\alpha)}(0), \ \ 0 < \alpha \leq 1 \qquad (12)$$

Proof:

Using the formula, and integration by parts, we get the following

$$E_\alpha\{f^{(\alpha)}(x)\} = v^\alpha F_\alpha\left(\frac{1}{v}\right)$$

$$= \frac{v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right) f^{(\alpha)}(t)(dt)^\alpha \qquad (13)$$

Fractional Elzaki Transform of some Functions

1. If $f(t) = E_\alpha(i^\alpha t^\alpha)$, we obtain

$$E_\alpha\{i^\alpha t^\alpha\}$$

$$= E_\alpha\left\{\sum_{k=0}^{\infty}\frac{i^{n\alpha}t^{n\alpha}}{\Gamma(1+n\alpha)}\right\} \qquad (14)$$

$$= \sum_{k}^{\infty} i^{k\alpha} v^{k\alpha+2\alpha}, \qquad (15)$$

And if $f(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}$, we get

$$E_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\sigma)}\right\} = v^{3\alpha} \qquad (16)$$

2. If $f(t) = E_\alpha(at^\alpha)$, then

$$E\{E_\alpha(at^\alpha)\}$$

$$= \frac{v\alpha}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right) E_\alpha(at^\alpha)(dt)^\alpha \qquad (17)$$

$$= \frac{v^\alpha}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha\left(-\frac{1-(av)^\alpha}{v^\alpha}t^\alpha\right)(dt)^\alpha \qquad (18)$$

$$= v^\alpha lim_{t\to\infty}\left[-\frac{v^\alpha}{1-(av)^\alpha}E_\alpha\left(-\frac{1-(av)^\alpha}{v^\alpha}t^\alpha\right)\right]_0^t \qquad (19)$$

$$= \frac{v^{2\alpha}}{1-(av)^\alpha} \qquad (20)$$

3. If $f(x) = \frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha(at)^\alpha$, by using the definition, we have

$$E_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha((at)^\alpha)\right\} = \frac{v^\alpha}{\Gamma(1+\alpha)}$$

$$\int_0^\infty E_\alpha\left(-\frac{t^\alpha}{v^\alpha}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}E_\alpha((at)^\alpha)(dt)^\alpha \qquad (21)$$

$$= \frac{v^\alpha}{\Gamma(1+\alpha)}$$

$$\int_0^\infty\left(-\frac{v^\alpha}{1-(av)^\alpha}E_\alpha\left(\frac{-1-(av)^\alpha}{v^\alpha}t^\alpha\right)\right)^{(\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}(dt)^\alpha \qquad (22)$$

$$= v^\alpha\lim_{t\to\infty}\left[-\frac{v^\alpha}{1-(av)^\alpha}E_\alpha\left(\frac{1-(av)^\alpha}{v^\alpha}t^\alpha\right)\frac{t^\alpha}{\Gamma(1+\alpha)}\right]_0^t \qquad (23)$$

$$+\frac{v^\alpha}{\Gamma(1+\alpha)}\lim_{t\to\infty}\int_0^t\frac{v^\alpha}{1-(av)^\alpha}E_\alpha\left(\frac{-(1-(av)^\alpha}{v^\alpha}t^\alpha\right)(dt)^\alpha \qquad (24)$$

$$= v^\alpha\lim_{t\to\infty}\left[-\frac{v^{2\alpha}}{(1-(av)^\alpha)^2}E_\alpha\left(-\frac{1-(av)^\alpha}{v^\alpha}t^\alpha\right)\right]_0^t \qquad (25)$$

Because

$\lim_{t\to\infty}\left[-\frac{v^\alpha}{1-(av)^\alpha}E_\alpha\left(\frac{1-(av)^\alpha}{v^\alpha}t^\alpha\right)\frac{t^\alpha}{\Gamma(1+\alpha)}\right]_0^t = 0$,

Therefore, we get

$$E_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\alpha)}E\_\alpha((ax)^\alpha)\right\}$$

$$= \frac{v^{3\alpha}}{(1-(av)^\alpha)^2} \qquad (26)$$

Adomian Decomposition Method

The Adomian Decomposition Method (ADM) is an analytical technique used for solving varieties of linear and nonlinear differential equations developed by George Adomian in the 1980s.

Consider a general nonlinear differential equation of the form:

$$L[u] + R[u] + N[u] = g(x), \qquad (27)$$

whereL is a linear operator,R is a remainder operator (usually involving lower-order derivatives),N is a nonlinear operator,g(x) is a known function,u(x) is the unknown function to be determined.

We express the solution of $u(x)$ as an infinite series as:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad (28)$$

where $u_n(x)$ are the components to be determined. Specifically, the nonlinear term $N[u]$ is decomposed using Adomian Polynomials defined as:

$$N\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} A_n, \qquad (29)$$

where A_n are the Adomian polynomials and are generated recursively.

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\Bigg|_{\lambda=0} \qquad (30)$$

The polynomials are generated using equation (33) . For a nonlinear function $N(u) = u^2$, the first few Adomian polynomials are:

$A_0 = u_0^2,$

$A_1 = 2u_0u_1,$

$A_2 = 2u_0u_2 + u_1^2,$

$A_3 = 2u_0u_3 + 2u_1u_2,$

$A_4 = 2u_0u_4 + 2u_1u_3 + u_2^2.$

These polynomials are obtained by substituting $\sum_{n=0}^{\infty} u_n$ into the nonlinear term and differentiating with respect to an auxiliary parameter λ .

Elzaki Adomian Decomposition Method (EADM)

Assume a general fractional partial differential equation of the form

$$D_t^\alpha u + Lu(x,t) + Nu(x,t) = f(x,t) \quad where \quad 0 \le \alpha \le 1 \quad and \quad 0 < t < 1 \qquad (31)$$

With an initial condition

$$u(x,0) = h(x) \qquad (32)$$

Applying the Elzaki Transform on equation (), the equation becomes

$$E\_\alpha_\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha} + Lu(x,t) + Nu(x,t)\right] = E_\alpha[f(x,t)] \qquad (33)$$

Applying the Linearity property of Elzaki transform

$$E_\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right] + E_\alpha[Lu(x,t) + Nu(x,t)] = E_\alpha[f(x,t)] \qquad (34)$$

$$E_\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right] = E_\alpha[f(x,t)] - E_\alpha[Lu(x,t) + Nu(x,t)] \qquad (35)$$

Applying the Differential Property of Elzaki Transform, the equation becomes

$$\frac{1}{v^\alpha} E\_\alpha[u(x,t)] - \sum_{n=0}^{m-1} v^{2-\alpha+n} u^{(n)}(x,0) = E_\alpha[f(x,t)] - E_\alpha[Lu(x,t) + Nu(x,t)], \qquad (36)$$

$$\frac{1}{v^\alpha} E_\alpha[u(x,t)] = \sum_{n=0}^{m-1} v^{2-\alpha+n} u^{(n)}(x,0) + E_\alpha[f(x,t)] - E_\alpha[Lu(x,t) + Nu(x,t)], \qquad (37)$$

multiply equation (31) by v^α

$$E_\alpha[u(x,t)] = \sum_{n=0}^{m-1} v^{2+n} u^{(n)}(x,0) + v^\alpha E_\alpha[f(x,t)] - v^\alpha E_\alpha[Lu(x,t) + Nu(x,t)], \qquad (38)$$

taking the Elzaki inverse of equation (39)

$$u(x,t) = E_\alpha^{-1}\left\{\sum_{n=0}^{m-1} v^{2+n} u^{(n)}(x,0) + v^\alpha E_\alpha[f(x,t)]\right\} - E_\alpha^{-1}\{v^\alpha E\_\alpha[Lu(x,t) + Nu(x,t)]\}, \qquad (40)$$

applying the iterative method

$$u(x,t) = \sum_{i=1}^{\infty} u_i(x,t), \qquad (41)$$

the nonlinear coefficient is written as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \qquad (42)$$

the Adomian polynomial A_n can be expressed as

$$A_0 = u_0 \frac{\partial u_0}{\partial x},$$

$$A_1 = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x},$$

$$A_2 = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x},$$

$$A_3 = u_0 \frac{\partial u_3}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_0}{\partial x},$$

the Adomian polynomial can be generally written as

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} \qquad (43)$$

Assume

$$H(x) \ = E_\alpha^{-1}\left\{\sum_{n=0}^{m-1} v^{2+n} u^{(n)}(x,0) + v^\alpha E\_\alpha[f(x,t)]\right\}$$

substituting equation (35) into (33), the equation becomes

$$u_{n+1}(x,t) = H(x) - E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[Lu(x,t) + \sum_{n=0}^{\infty} A_n\right]\right\}, \qquad (44)$$

$$u_{n+1}(x,t) = H(x) - E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[Lu(x,t) + \sum_{n=0}^{\infty} A_n\right]\right\}, \qquad (45)$$

subsequently, the iteration can be written as

$$u_0 = H(x)$$

$$u_1 = -E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[Lu_0(x,t) + \sum_{n=0}^{\infty} A_0\right]\right\}, \quad when \quad n = 0 \qquad (46)$$

$$u_2 = -E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[Lu_1(x,t) + \sum_{n=0}^{\infty} A_1\right]\right\}, \quad when \quad n = 1 \qquad (47)$$

$$u_3 = -E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[Lu_2(x,t) + \sum_{n=0}^{\infty} A_2\right]\right\}, \quad when \quad n = 2 \qquad (48)$$

Result And Discussion

In this section, examples on fractional damped Burger's Equation is solved.

Example 1. Examine the following nonlinear Fractional damped Burger Equation (Liu, 2023)

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{u\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + \xi u = 0, \qquad 0 < \alpha \leq 1, \qquad t \in (0,T], \quad x \in R \qquad (49)$$

$$u(x,0) = \xi x \qquad (50)$$

when $\alpha = 1$, the exact solution with initial condition corresponding to

$$u(x,t) = \frac{\xi x}{2e^{\xi t} - 1} \qquad (51)$$

equation (42) can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u, \qquad (52)$$

taking the Elzaki transform of equation (45)

$$E_\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right] = E_\alpha\left[\frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u\right], \qquad (53)$$

$$\frac{1}{v^{\alpha}}E_{\alpha}[u(x,t)] - \sum_{n=0}^{m-1} v^{2-\alpha+n}u^{(n)}(x,0) = E_{\alpha}\left[\frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u\right], \qquad (54)$$

$$\frac{1}{v^{\alpha}}E_{\alpha}[u(x,t)] = \sum_{n=0}^{m-1} v^{2-\alpha+n}u^{(n)}(x,0) + E_{\alpha}\left[\frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u\right], \qquad (55)$$

multiply equation (48) by v^{α}

$$E_{\alpha}[u(x,t)] = \sum_{n=0}^{m-1} v^{2+n}u^{(n)}(x,0) + v^{\alpha}E_{\alpha}\left[\frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u\right], \qquad (56)$$

taking the Elzaki inverse of equation (49)

$$u(x,t) = E_{\alpha}^{-1}\left\{\sum_{n=0}^{m-1} v^{2+n}u^{(n)}(x,0)\right\} + E_{\alpha}^{-1}\left\{v^{\alpha}E_{\alpha}\left[\frac{\partial^2 u}{\partial x^2} - \frac{u\partial u}{\partial x} - \xi u\right]\right\}, \qquad (57)$$

applying the iterative method

$$u(x,t) = \sum_{i=1}^{\infty} u_i(x,t),$$

the nonlinear coefficient is written as

$$\frac{u\partial u}{\partial x} = \sum_{n=0}^{\infty} A_n \qquad (58)$$

the Adomian polynomial A_n can be expressed as

$$A_0 = u_0\frac{\partial u_0}{\partial x},$$

$$A_1 = u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x},$$

$$A_2 = u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x},$$

$$A_3 = u_0\frac{\partial u_3}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_0}{\partial x},$$

The Adomian polynomial can be generally written as

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \qquad (59)$$

Substituting equation (51) into (50), the equation becomes

$$u_{n+1}(x,t) = E_{\alpha}^{-1}\{\xi x\} + E\_\alpha{}^{-1}\left\{v^{\alpha}E_{\alpha}\left[\frac{\partial^2}{\partial x^2}\sum_{0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n - \xi\sum_{n=0}^{\infty} u_n\right]\right\}, \qquad (60)$$

$$u_{n+1}(x,t) = \xi x + E\_\alpha{}^{-1}\left\{v^{\alpha}E_{\alpha}\left[\frac{\partial^2}{\partial x^2}\sum_{0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n - \xi\sum_{n=0}^{\infty} u_n\right]\right\}, \qquad (61)$$

Subsequently, the iteration can be written as

$$u_0 = \xi x \qquad (62)$$

When $n = 0$

$$u_1 = E_{\alpha}^{-1}\left\{v^{\alpha}E_{\alpha}\left[\frac{\partial^2 u_0}{\partial x^2} - A_0 - \xi u_0\right]\right\}$$

$$= E_{\alpha}^{-1}\left\{v^{\alpha}E_{\alpha}\left[u_0\frac{\partial u_0}{\partial x} - \xi u_0\right]\right\}$$

$$= E_{\alpha}^{-1}\{v^{\alpha}E_{\alpha}[-\xi x \times \xi - \xi \times \xi x]\}$$

$$= E_{\alpha}^{-1}\{v^{\alpha}E_{\alpha}[-2\xi^2 x]\}$$

$$= -2\xi^2 x E_{\alpha}^{-1}\{v^{\alpha+2\alpha}\}$$

$$= -2\xi^2 x E_\alpha^{-1}\{v^{3\alpha}\}$$

$$u_1 = -\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)} \qquad (63)$$

When $n = 1$

$$u_2 = E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - A_1 - \xi u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - \left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) - \xi u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - \xi u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2}{\partial x^2}\left(-\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right) - \xi x\frac{\partial}{\partial x}\left(-\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right) - \left(-\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right)\frac{\partial(\xi x)}{\partial x} - \xi\left(-\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right)\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\xi x\left(\frac{2\xi^2 t^\alpha}{\Gamma(\alpha+1)}\right) + \left(\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right)\xi + \left(\frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)}\right)\xi\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{2\xi^3 x t^\alpha}{\Gamma(\alpha+1)} + \frac{2\xi^3 x t^\alpha}{\Gamma(\alpha+1)} + \frac{2\xi^3 x t^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{6\xi^3 x t^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E\_\alpha^{-1}\left\{v^\alpha\left[\frac{6\xi^3 x v^{3\alpha}}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E\_\alpha^{-1}\{6\xi^3 x v^{4\alpha}\}$$

$$u_2 = \frac{6\xi^3 x t^2}{\Gamma(2\alpha+1)} \qquad (64)$$

The Approximate solution can be written as

$$u(x,t) = \xi x - \frac{2\xi^2 x t^\alpha}{\Gamma(\alpha+1)} + \frac{6\xi^3 x t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots + \qquad (65)$$

Results obtained is the same as result obtained in Liu (2023).

Table 1: Behaviour of example 1 with different values of alpha

x						.0	Liu (2023)
						-7	-7
						652	-5.1652
						425	-3.7425
						533	-2.6533
						306	-1.8306
						218	-1.218
						769	-0.769
						453	-0.4453
						165	-0.2165
						585	-0.0585
						474	0.0474
						151	0.1151
						555	0.1555
						769	0.1769
						863	0.1863
						903	0.1903
0.68	1.5116	0.8679	0.508	0.3233	0.2352	0.1963	0.1963
0.79	2.3032	1.3551	0.7815	0.4626	0.2964	0.2135	0.2135
0.89	3.3601	2.0458	1.1935	0.6867	0.4046	0.254	0.254
1	4.7252	2.9871	1.7873	1.0299	0.584	0.3333	0.3333

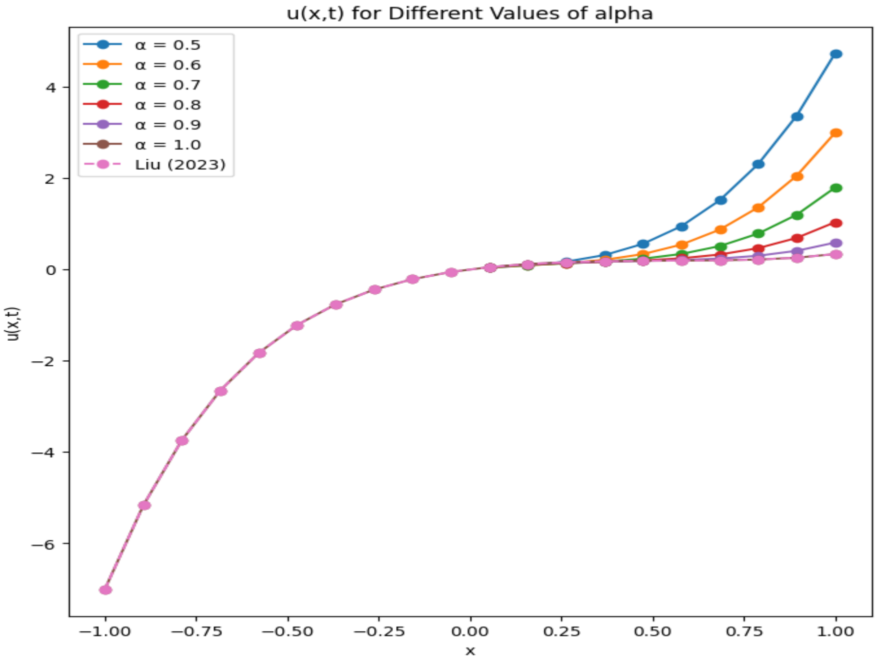

Figure 1 Behaviour of example 1 with different values of alpha

Example 2. Examine the following nonlinear Fractional damped Burger Equation (Abd AL-Hussein, 2018)

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{u\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}+2u=0, \qquad 0<\alpha \leq 1, \qquad t\in(0,T], \quad x\in R \qquad (66)$$

$$u(x,0)=x \qquad (67)$$

when $\alpha=1$, the exact solution with initial condition corresponding to

$$u(x,t)=\frac{\xi x}{2e^{\xi t}-1} \qquad (68)$$

equation (59) can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u, \qquad (69)$$

taking the Elzaki transform of equation (61)

$$E_\alpha\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right]=E_\alpha\left[\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u\right], \qquad (70)$$

$$\frac{1}{v^\alpha}E_\alpha[u(x,t)]-\sum_{n=0}^{m-1}v^{2-\alpha+n}u^{(n)}(x,0)=E_\alpha\left[\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u\right], \qquad (71)$$

$$\frac{1}{v^\alpha}E_\alpha[u(x,t)]=\sum_{n=0}^{m-1}v^{2-\alpha+n}u^{(n)}(x,0)+E_\alpha\left[\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u\right], \qquad (72)$$

multiply equation (64) by v^α

$$E_\alpha[u(x,t)]=\sum_{n=0}^{m-1}v^{2+n}u^{(n)}(x,0)+v^\alpha E_\alpha\left[\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u\right], \qquad (73)$$

taking the Elzaki inverse of equation (74)

$$u(x,t)=E_\alpha^{-1}\left\{\sum_{n=0}^{m-1}v^{2+n}u^{(n)}(x,0)\right\}+E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u}{\partial x^2}-\frac{u\partial u}{\partial x}-2u\right]\right\}, \qquad (75)$$

applying the iterative method

$$u(x,t)=\sum_{i=1}^{\infty}u_i(x,t) \qquad (76)$$

the nonlinear coefficient is written as

$$\frac{u\partial u}{\partial x}=\sum_{n=0}^{\infty}A_n \qquad (77)$$

the Adomian polynomial A_n can be expressed as

$$A_0=u_0\frac{\partial u_0}{\partial x},$$

$$A_1=u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x},$$

$$A_2=u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_0}{\partial x},$$

$$A_3=u_0\frac{\partial u_3}{\partial x}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_1}{\partial x}+u_3\frac{\partial u_0}{\partial x},$$

the Adomian polynomial can be generally written as

$$A_n(u_0,u_1,\dots,u_n)=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \qquad (78)$$

substituting equation (12) into (9), the equation becomes

$$u_{n+1}(x,t)=E_\alpha^{-1}\{x\}+E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2}{\partial x^2}\sum_{0}^{\infty}u_n-\sum_{n=0}^{\infty}A_n-2\sum_{n=0}^{\infty}u_n\right]\right\}, \qquad (79)$$

$$u_{n+1}(x,t) = x + E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2}{\partial x^2}\sum_{0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n - 2\sum_{n=0}^{\infty} u_n\right]\right\}, \qquad (80)$$

subsequently, the iteration can be written as

$$u_0 = x \qquad (81)$$

when $n = 0$

$$u_1 = E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_0}{\partial x^2} - A_0 - 2u_0\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[u_0\frac{\partial u_0}{\partial x} - 2u_0\right]\right\}$$

$$= E_\alpha^{-1}\{v^\alpha E_\alpha[-x \times 1 - 2 \times x]\}$$

$$= E_\alpha^{-1}\{v^\alpha E_\alpha[-3x]\}$$

$$= -3xE_\alpha^{-1}\{v^{\alpha+2\alpha}\}$$

$$= -3xE_\alpha^{-1}\{v^{3\alpha}\}$$

$$u_1 = -\frac{3xt^\alpha}{\Gamma(\alpha+1)} \qquad (82)$$

When $n = 1$

$$u_2 = E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - A_1 - 2u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - \left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) - 2u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2 u_1}{\partial x^2} - u_0\frac{\partial u_1}{\partial x} - u_1\frac{\partial u_0}{\partial x} - 2u_1\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{\partial^2}{\partial x^2}\left(-\frac{3xt^\alpha}{\Gamma(\alpha+1)}\right) - x\frac{\partial}{\partial x}\left(-\frac{3xt^\alpha}{\Gamma(\alpha+1)}\right) - \left(-\frac{3xt^\alpha}{\Gamma(\alpha+1)}\right)\frac{\partial(x)}{\partial x} - 2\left(-\frac{3xt^\alpha}{\Gamma(\alpha+1)}\right)\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[x\left(\frac{3t^\alpha}{\Gamma(\alpha+1)}\right) + \left(\frac{3xt^\alpha}{\Gamma(\alpha+1)}\right) + \frac{6xt^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{3xt^\alpha}{\Gamma(\alpha+1)} + \frac{3xt^\alpha}{\Gamma(\alpha+1)} + \frac{6xt^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E_\alpha^{-1}\left\{v^\alpha E_\alpha\left[\frac{12xt^\alpha}{\Gamma(\alpha+1)}\right]\right\}$$

$$= E\_\alpha^{-1}\{v^\alpha[12xv^{3\alpha}]\}$$

$$= E\_\alpha^{-1}\{12xv^{4\alpha}\}$$

$$u_2 = \frac{12xt^{2\alpha}}{\Gamma(2\alpha+1)} \qquad (83)$$

the Approximate solution can be written as

$$u(x,t) = x - \frac{3xt^\alpha}{\Gamma(\alpha+1)} + \frac{12xt^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots + \qquad (84)$$

the Exact solution when $\alpha = 1$ can be written as

$$u(x,t) = \frac{x}{2e^x - 1} \qquad (85)$$

Table 2 Behaviour of Example 2 with different values of alpha

x	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	Abd (2018)
-1	-0.1732	-0.3041	-0.4276	-0.5385	-0.6343	-0.7145	-0.7145
-0.89	-0.0425	-0.1454	-0.2503	-0.3512	-0.4442	-0.5269	-0.5269
-0.79	0.0368	-0.0394	-0.1237	-0.2094	-0.2926	-0.3704	-0.3704

-0.68	0.0823	0.0294	-0.0348	-0.104	-0.1743	-0.2429	-0.2429
-0.58	0.1034	0.0703	0.0241	-0.0289	-0.0853	-0.1424	-0.1424
-0.47	0.1066	0.0893	0.0585	0.0203	-0.0224	-0.0673	-0.0673
-0.37	0.0964	0.0906	0.0724	0.047	0.017	-0.0159	-0.0159
-0.26	0.0762	0.0776	0.0689	0.054	0.0351	0.0135	0.0135
-0.16	0.0489	0.0532	0.0505	0.0437	0.034	0.0223	0.0223
-0.05	0.017	0.0196	0.0196	0.018	0.0154	0.0119	0.0119
0.05	-0.0174	-0.0211	-0.022	-0.0212	-0.0191	-0.0162	-0.0162
0.16	-0.0522	-0.0669	-0.0725	-0.0723	-0.068	-0.0609	-0.0609
0.26	-0.0856	-0.1162	-0.1303	-0.1338	-0.13	-0.1209	-0.1209
0.37	-0.116	-0.1671	-0.1939	-0.2042	-0.2037	-0.195	-0.195
0.47	-0.1415	-0.2181	-0.2616	-0.2824	-0.2878	-0.2822	-0.2822
0.58	-0.1606	-0.2675	-0.3321	-0.3669	-0.3813	-0.3812	-0.3812
0.68	-0.1717	-0.3138	-0.4039	-0.4564	-0.4829	-0.4911	-0.4911
0.79	-0.1733	-0.3553	-0.4756	-0.5496	-0.5915	-0.6107	-0.6107
0.89	-0.1639	-0.3906	-0.5456	-0.6453	-0.7059	-0.739	-0.739
1	-0.1419	-0.4179	-0.6125	-0.7421	-0.8249	-0.875	-0.875

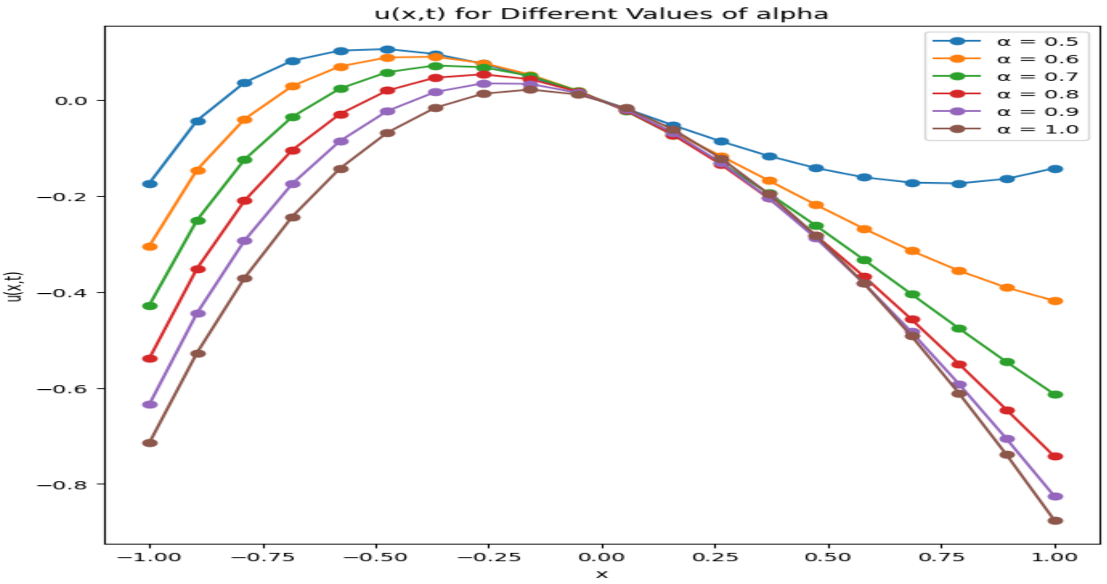

Figure 2: Behaviour of Example 2 with different values of alpha

Result obtained is the same as result gotten from Abd (2018)

Conclusion

The solution technique used, the Elzaki Adomian Decomposition Method (LFEADM), effectively handles fractional derivatives and nonlinearities by combining the benefits of the Elzaki Transform and the Adomian Decomposition Method (ADM). The fractional Burgers equation is reduced to a a less complex equation by using EADM, and the nonlinear components are broken down into a set of approximations through the use of Adomian polynomials. This hybrid method works especially well for damped system analysis because it offers a strong foundation for solving nonlinear fractional differential equations. The influence of the damping factor ξ on the Fractional Damped Burgers Equation indicates discrete behavioral differences in the system. The damping effect is significant and causes wave propagation to decline quickly showing the most suppression. .

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